Born-Infeld corrections to Coulombian interactions

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Two-dimensional Born-Infeld electrostatic fields behaving as the superposition of two pointlike charges in the linearized (Maxwellian) limit are investigated by means of a nonholomorphic mapping of the complex plane. The changes in the Coulombian interaction between two charges in Born-Infeld theory are computed. Remarkably, the force between equal charges goes to zero as they approach each other.

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I. INTRODUCTION

When forces between *charges* are considered in nonlinear theories, the picture of the field due to a charge acting on another charge is no longer applicable. Since the superposition principle is not feasible, a multiple-charge configuration has to be analyzed as a new problem instead of the mere addition of already known solutions. Even the expression "multiple-charge configuration" calls for an explanation. For those theories behaving as linear theories in the weak-field regime, a multiple-charge configuration can be defined as a solution of the nonlinear field equations going to a superposition of individual (linear) charges at infinity. Given a static multiple-charge solution of some nonlinear theory, the force on a charge can be worked out by computing the flux of the stress tensor through a surface surrounding the charge. Since the stress tensor is divergenceless, the resulting force will be different from zero only at those points where the stress tensor is singular, which provides a way of localizing the charges.

This paper is aimed at solving two-dimensional static configurations of two charges in Born-Infeld nonlinear electrodynamics, and computing the interaction strength. In Sec. II we summarize the Born-Infeld theory. In Sec. III we characterize the two-dimensional electrostatic solutions by means of a nonholomorphic complex transformation. In Sec. IV we obtain the repulsive and attractive interactions between equal and opposite charges. We compute corrections to the Coulombian interaction for distant charges, and show that the repulsive force vanishes when equal charges approach each other. The conclusions are displayed in Sec. V.

II. BORN-INFELD ELECTRODYNAMICS

Born-Infeld electrodynamics is a nonlinear theory whose initial objective was to render finite the self-energy of a pointlike charge. In Born-Infeld electrostatics the electric field **E** due to a pointlike charge does not diverge but goes to a finite value b at the charge position. The energy-momentum tensor still diverges at the charge position but the integral of the energy density becomes finite. The fundamental constant b is an upper bound for the fields, and regulates the transition to the weak-field regime: for fields much smaller than b the theory behaves like Maxwell electromagnetism. By healing the field of singularities, Born and Infeld thought that the theory could be regarded from a unitary standpoint: the only physical entity would be the field, whereas the charges would be just a part of the field [1–15]. They even believed that the solutions would contain some essential features of the charge dynamics, which is by no means true since the theory allows for static multiple-charge solutions.

Born-Infeld electrodynamics possesses outstanding physical properties: it and Maxwell theory are the only spin-1 field theories having causal propagation [6,7] and absence of birefringence [6,8]. Although concrete solutions for propagating Born-Infeld electromagnetic waves are not sufficiently known—apart from trivial free-waves solutions—solutions for waves propagating in static background fields and waveguides have been recently obtained [9,10]. The renewed interest in Born-Infeld theory can be traced to its emergence in the study of strings and branes: loop calculations for open superstrings lead to a Born-Infeld-type low-energy action [11–13]. Nowadays Born-Infeld-like Lagrangians have been proposed for quintessential matter models and inflation [14], and also for alternative theories of gravity [15]. Born-Infeld charges coupled with gravity have been investigated in attempts to remove geometrical singularities of charged black holes [16].

The Born-Infeld Lagrangian density for the electromagnetic field $F_{ij}=\partial_i A_j-\partial_j A_i$ is [2]

$$L_{\rm BI} = -\frac{1}{4\pi c}\left[\sqrt{|\det(bg_{ij}+F_{ij})|}-\sqrt{|\det(bg_{ij})|}\right]$$
$$= \frac{\sqrt{-g}}{4\pi c}b^2\left(1-\sqrt{1+b^{-2}2S-b^{-4}P^2}\right), \qquad (1)$$

where S and P are the scalar and pseudoscalar invariants

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$$S = \frac{1}{4}F_{ij}F^{ij} = \frac{1}{2}(B^2 - E^2),$$
$$P = \frac{1}{4}\,{}^*F_{ij}F^{ij} = \mathbf{E}\cdot\mathbf{B}, \tag{2}$$

and b is a new universal constant with units of field, which plays the role of an upper bound for the electrostatic field. The term $\sqrt{|\det(bg_{ij})|}$ in (1) makes L_{BI} vanish when the electromagnetic field vanishes. L_{BI} goes to the Maxwell Lagrangian density in the limit $b\to\infty$. By defining the two-form

$$\mathcal{F}_{ij} = \frac{F_{ij} - b^{-2}P\,{}^*F_{ij}}{\sqrt{1 + b^{-2}2S - b^{-4}P^2}}, \tag{3}$$

we can write the Euler-Lagrange equations coming from L_{BI} as

$$d\,{}^*\mathcal{F} = 0. \tag{4}$$

These equations are supplemented with the identities $dF=0$ (i.e., $F_{[jk,i]}=0$), since the field is an exact two-form ($F=dA$). The energy-momentum tensor is

$$T_{ij} = -\frac{1}{4\pi}F_{ik}\mathcal{F}^k_j - \frac{b^2}{4\pi}g_{ij}(1 - \sqrt{1 + b^{-2}2S - b^{-4}P^2}), \tag{5}$$

which verifies energy-momentum conservation,

$$T^l_{k;l} = 0, \tag{6}$$

at all the places where T_{ij} is nonsingular. For a pointlike charge Q, $\mathcal{F}=Qr^{-2}dt\wedge dr$ diverges at the charge position but F is finite. Thus, although the energy density still diverges at the charge position, the integrated energy turns out to be finite.

Owing to the nonlinear character of Eq. (4) for the field F, it is hard to find explicit solutions in an analytic way, apart from highly symmetric configurations. In more general cases, the solutions are displayed only in an implicit form [17,18].

III. TWO-DIMENSIONAL ELECTROSTATIC SOLUTIONS

Two-dimensional electrostatic solutions, together with conditions guaranteeing uniqueness of the solutions, have been worked out in a rather cumbersome parametric way by resorting to the relationship between minimal surface equations and the Born-Infeld electrostatic problem [19,20]. Recently, two-dimensional electrostatic solutions in Euclidean space have been obtained by using a nonholomorphic transformation of the complex plane. This method has been used for working out the field lines and self-energies of pointlike two-dimensional multipoles [21]. The method finds a coordinate transformation in the plane, $(x,y)\to(u,v)$, such that (u,v) are orthogonal coordinates and $u(x,y)$ is the potential for the Born-Infeld electrostatic field: $\mathbf{E}=-\boldsymbol{\nabla}u$ (then the coordinate lines are equipotential and field lines, respectively). The equation to be satisfied by $u(x,y)$ is the one resulting from Eq. (4) when the field F is chosen to be the exact two-form $F=du\wedge dt$, corresponding to the electrostatic potential $A=u(x,y)dt$. In current language, the equation is [22]

$$\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}u(x,y)}{\sqrt{1 - b^{-2}|\boldsymbol{\nabla}u(x,y)|^2}}\right) = 0, \tag{7}$$

which becomes the Laplace equation when $b\to\infty$. Actually this equation can also be obtained by directly suppressing the time coordinate t and the third spatial coordinate in the Lagrangian. Thus, in two dimensions the (static) field is the exact one-form $F=du$. In addition, ${}^*\mathcal{F}$ in Eq. (4) is the one-form ${}^*\mathcal{F}={}^*F/\sqrt{1-b^{-2}E^2}$.

In terms of complex numbers $z=x+iy$ and $w=u+iv$, the coordinate transformation $(x,y)\to(u,v)$ can be regarded as $z\to z(w,\bar{w})$ or

$$dz = p(w,\bar{w})dw + q(w,\bar{w})d\bar{w} \tag{8}$$

(the overbar means the complex conjugate). The integrability condition for Eq. (8) requires that

$$\bar{\partial}p(w,\bar{w}) = \partial q(w,\bar{w}), \tag{9}$$

where ∂ and $\bar{\partial}$ are exterior derivatives with respect to w and $\bar{w}$ (Dolbeault operators). To obtain the Euclidean metric in (u,v) coordinates, one can write

$$dx^2 + dy^2 = dz\,d\bar{z} = (|p|^2 + |q|^2)|dw|^2 + 2\,\mathrm{Re}(p\bar{q}\,dw^2). \tag{10}$$

To get orthogonal coordinates (u,v), the terms containing $du\,dv$ must be taken out of the quadratic form (10). Then

$$\mathrm{Im}(p\bar{q}) = 0. \tag{11}$$

The solution for the one-form dw can be obtained from Eq. (8) and its complex conjugate:

$$dw = \frac{\bar{p}\,dz - q\,d\bar{z}}{|p|^2 - |q|^2}. \tag{12}$$

In two-dimensional Euclidean space we have ${}^*dz=i\,dz$; then

$${}^*dw = i\frac{\bar{p}\,dz + q\,d\bar{z}}{|p|^2 - |q|^2}. \tag{13}$$

In order that the field ${}^*\mathcal{F}=\mathrm{Re}({}^*dw)/\sqrt{1-b^{-2}E^2}$ satisfies Eq. (4), we have to properly choose the functions p and q. Let us try the choice $q=0$. Then, the integrability condition (9) implies that $p=p(w)$ (w is a holomorphic function of z). In this case ${}^*dw=i\,dw$ is an exact one-form. So Eq. (4) is satisfied only if $b\to\infty$ [because the result will be ${}^*\mathcal{F}={}^*F=\mathrm{Re}({}^*dw)$]. This is the case of the Coulombian field (it is well known that holomorphic functions provide solutions for the Laplace equation in two dimensions). The Coulombian solution is

$$\mathbf{E} = E_x + iE_y = -\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = -\frac{1}{\overline{p(w)}}. \tag{14}$$

Let us try a choice of p and q leading to $q=0$ when $b\to\infty$. We guess the choice [21]

$$p(w)\overline{q(\bar{w})} = \frac{1}{4b^2} \tag{15}$$

[notice that the integrability and orthonormality conditions (9) and (11) are satisfied]. Then

$$du = \mathrm{Re}(w) = 4b^2 \frac{\mathrm{Re}(p)dx + \mathrm{Im}(p)dy}{4b^2|p|^2 + 1}, \tag{16}$$

$$dv = \mathrm{Im}(w) = 4b^2 \frac{-\mathrm{Im}(p)dx + \mathrm{Re}(p)dy}{4b^2|p|^2 - 1}. \tag{17}$$

Therefore

$$\mathbf{E} = E_x + iE_y = -\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = -\frac{4b^2 p(w)}{4b^2|p(w)|^2 + 1} \tag{18}$$

and

$$\sqrt{1 - \frac{E^2}{b^2}} = \pm \frac{4b^2|p(w)|^2 - 1}{4b^2|p(w)|^2 + 1}. \tag{19}$$

These results imply that ${}^*\mathcal{F}$ is an exact one-form. In fact

$${}^*\mathcal{F} = \frac{{}^*du}{\sqrt{1 - b^{-2}E^2}} = \pm i\, dv. \tag{20}$$

Therefore, Eq. (4) is satisfied.

Equation (18) shows the way to generate a Born-Infeld field behaving in the weak-field limit (far from the charges) like a given Coulombian field: choose the holomorphic function $p(w)$ associated with the Coulombian field (14) and replace it in (18). However, as the field should be expressed as a function of z or (x,y) instead of w or (u,v), we have to take into account that the relation between z and w is no longer the Coulombian relation; according to Eq. (8) the relation now is

$$dz = p(w)dw + \frac{1}{4b^2 p(w)} d\bar{w}. \tag{21}$$

This equation amounts to a nonholomorphic relation between z and w. The electrostatic potential satisfying the Born-Infeld equation (7) is $u(x,y) = \mathrm{Re}[w(z,\bar{z})]$. The function $p(w)$ in Eq. (21) plays the role of a Coulombian seed to obtain the Born-Infeld potential.

Equation (18) shows that $|\mathbf{E}|$ does not diverge but attains its maximum value b at the points where

$$|p(w)| = \frac{1}{2b}. \tag{22}$$

According to Eq. (15), at these points $|p| = |q|$; so they are singular points of the coordinate change (12) and (13). The relation (22) describes the curve where the energy-momentum tensor (5) is singular, because the vanishing of (19) implies that $\mathcal{F}$ diverges. This curve is then the charge location. If the curve $|p(w)| = (2b)^{-1}$ is closed, then it separates two different regions in the complex plane: (i) $|p(w)| > (2b)^{-1}$ and (ii) $|p(w)| < (2b)^{-1}$. Only the first region can realize the Coulombian limit $E/b \to 0$. Since the Born-Infeld field (18) should go to the Coulombian field (14) at infinity, the region $|p(w)| > (2b)^{-1}$ corresponds to the exterior of the charge distribution. The curves where $|p(w)| = (2b)^{-1}$ have been studied in Ref. [21] for the configurations associated with Coulombian multipoles. In these cases the curves are closed and turn out to be epicycloids whose sizes are determined by b and the multipolar moment. One can say that the Born-Infeld field smooths singularities in two different ways: on one hand it smooths the divergence of the energy-momentum tensor in order that the self-energy is finite; on the other hand the pointlike character of the Coulombian multipoles is spread to the surface (in this case a curve) where the field reaches the upper bound b.

IV. FORCE BETWEEN TWO-DIMENSIONAL MONOPOLES

We are going to study the electrostatic Born-Infeld configuration corresponding to two pointlike monopoles separated by a distance d in the Coulombian limit. The interaction force between charges will result from the momentum flux through a closed surface S containing one of the charges. We will choose the x axis along the line joining the charges, and the origin of coordinates at the intermediate point. Since the symmetry dictates that the force is directed along the x axis, the involved momentum flux is $T^{xj}dS_j$. As usual, we choose S as the surface formed by the y axis, and a semicircumference of infinite radius centered at the origin of coordinates; on this last surface the flux is null. Thus

$$F^x = -\int_{y\ \mathrm{axis}} T^{xx} n_x dS = \frac{b^2}{4\pi}\int_{-\infty}^{+\infty}\left[1 - \left(1 - \frac{E^2}{b^2}\right)^{1/2} - \frac{E_x^2}{b^2}\left(1 - \frac{E^2}{b^2}\right)^{-1/2}\right]_{x=0} dy. \tag{23}$$

For repulsive interactions between equal charges, $E_x = 0$ ($v = \mathrm{const}$) and $dy = (\partial y/\partial u)du$ at $x=0$. Thus, according to Eqs. (18), (19), and (21) the force is

$$F^x = \frac{1}{8\pi}\int_{y\ \mathrm{axis}} \frac{\mathrm{Im}[p(w)]}{|p(w)|^2} du. \tag{24}$$

For attractive interactions between opposite charges, $E_x = \pm E$ ($u = \mathrm{const}$) and $dy = (\partial y/\partial v)dv$ at $x=0$. Then the force is

$$F^x = \frac{1}{8\pi}\int_{y\ \mathrm{axis}} \frac{\mathrm{Re}[p(w)]}{|p(w)|^2} dv. \tag{25}$$

A. Equal charges

The well-known Coulombian potential $u(x,y)$ for the repulsive configuration of two equal charges λ at a distance d can be written as $u(x,y) = \mathrm{Re}[w(z)]$, where $w(z)$ is the holomorphic function

$$w = -2\lambda \ln\left[\left(\frac{2z}{d}\right)^2 - 1\right]. \tag{26}$$

In Eq. (26) a proper choice of the integration constant was made in order that the Coulombian potential be zero at the origin. Relation (26) can be inverted to obtain the holomorphic function p characterizing the Coulombian mapping:

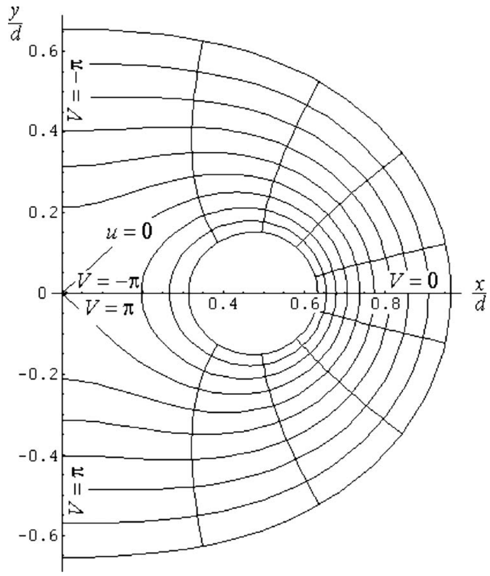

FIG. 1. Equipotential and field lines for the Coulombian equal charges (right semispace). V stands for $v/(2\lambda)$.

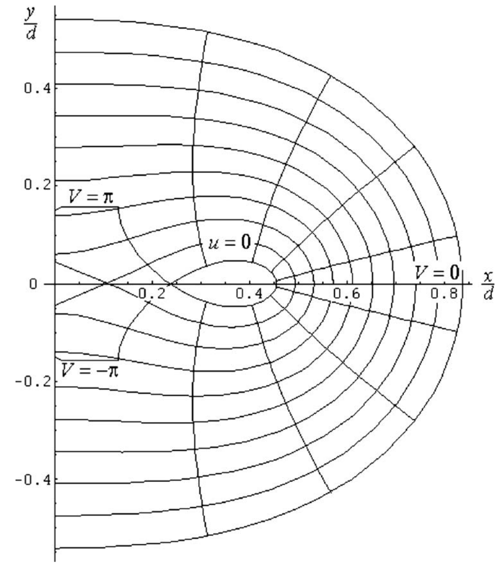

FIG. 2. Multivalued Born-Infeld mapping (29) for $-\pi < v/(2\lambda) < \pi$ ($\alpha^2=40$). In the figure V stands for $v/(2\lambda)$.

$$z = \pm\frac{d}{2}\sqrt{\exp\left(-\frac{w}{2\lambda}\right)+1}, \tag{27}$$

where $\pm$ alludes to the $x>0$ and $x<0$ regions. By differentiating Eq. (27) one obtains

$$p(w) = \mp\frac{d}{8\lambda}\frac{\exp\left(-\frac{w}{2\lambda}\right)}{\sqrt{\exp\left(-\frac{w}{2\lambda}\right)+1}}. \tag{28}$$

By substituting this in Eq. (21) and then integrating, one obtains the mapping leading to the Born-Infeld complex potential $w(z,\bar{z})$:

$$z = \pm\frac{d}{2}\left(\sqrt{\exp\left(-\frac{w}{2\lambda}\right)+1} - \frac{8}{\alpha^2}\left\{\exp\left(\frac{\bar{w}}{2\lambda}\right)\sqrt{\exp\left(-\frac{\bar{w}}{2\lambda}\right)+1} + \frac{\bar{w}}{4\lambda} + \ln\left[1+\sqrt{\exp\left(-\frac{\bar{w}}{2\lambda}\right)+1}\right]\right\}\right), \tag{29}$$

where α is the nondimensional parameter $\alpha=bd/\lambda$. In Eq. (29) one recognizes the holomorphic (Coulombian) seed (27) and the antiholomorphic Born-Infeld correction. Unlike the Coulombian case (27), Eq. (29) does not provide a unique electrostatic potential u to each point of the complex plane. In particular, while the Coulombian mapping (27) is periodic in the v coordinate, the Born-Infeld mapping (29) fails to be periodic because of the presence of the linear term $\bar{w}/(4\lambda)$. Figure 1 shows the u,v lines for the Coulombian potential (27). The equipotential $u=0$ passes through the coordinate origin. The lines $v/(2\lambda)=\pm\pi$ coincide with the y axis ($u>0$) and the part of the x axis joining the charges ($u<0$). The line $v=0$ is the piece of the x axis going from the charges to infinity. Figure 2 shows the u,v lines resulting from the Born-Infeld mapping (29) for $-\pi<v/(2\lambda)<\pi$. We still have $v/(2\lambda)=\pm\pi$ at the y axis; however, as a consequence of the loss of the periodicity, the lines $v/(2\lambda)=\pm\pi$ do not end at the x axis but cross it, so giving rise to a multivalued figure for the potential $u(x,y)$. To have a single-valued potential, the u-v domain of the mapping (29) should be restricted by cutting it at the x axis (owing to this branch cut, the field is not continuous along the line joining the charges). As in the Coulombian case, the line $v=0$ is the piece of the x axis from the charges to infinity. The function $p(w)$ is real on the line $v=0$ [see Eq. (28)], and attains its maximum value $1/(2b)$ at the charges, where the potential is

$$\exp\left(-\frac{u_{\max}}{2\lambda}\right) = \frac{4}{\alpha^2}\left(2+\sqrt{4+\alpha^2}\right). \tag{30}$$

Since the equipotential lines surround the charges, the value (30) is the maximum value for the Born-Infeld potential. Notice that, by solving the equation $|p(w)|=1/(2b)$ for any value of v, one obtains the curve $u(v)$ described by the relation

$$\left(\frac{\alpha}{4}\right)^4\exp\left(-\frac{2u}{\lambda}\right) - \exp\left(-\frac{u}{\lambda}\right) - 2\cos\left(\frac{v}{2\lambda}\right)\exp\left(-\frac{u}{2\lambda}\right) = 1. \tag{31}$$

This curve decomposes into two parts at each side of the y axis, displaying cusps on the x axis at $|x|<d/2$ (see Fig. 3). In spite of the appearance, the charge configuration is not spread on the lobes of Fig. 3, but concentrates at the cusps. In fact, Eq. (31) can be satisfied only for $u\geq u_{\max}$. Thus the only admissible solution of Eq. (31) is $(u=u_{\max}, v=0)$, i.e., the positions of the pointlike charges. The remainder of the lobes is cut because the mapping domain is restricted to have a single-valued potential matching the Coulombian potential at infinity.

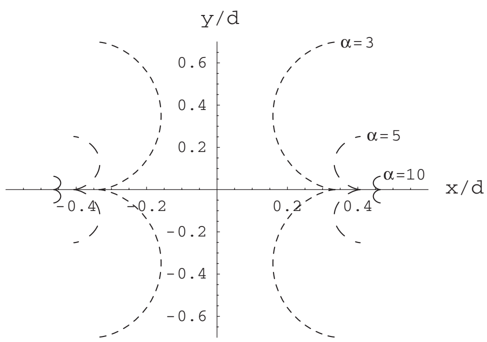

FIG. 3. Curves $|p(w)|=(2b)^{-1}$ characterized by $\alpha = bd/\lambda$ for the configuration of two equal charges. Only the cusps, but not the lobes, belong to the branch of $w(z)$ under consideration.

In order to compute the force (24) we will take into account that the points on the y axis satisfy

$$\frac{u}{2\lambda} < 0 \quad \text{and} \quad \frac{v}{2\lambda} = \pm \pi. \tag{32}$$

Then, according to Eq. (29), the positive y semiaxis can be parametrized by defining a parameter t such that $\cos(t/2) = \exp[u/(4\lambda)]$; thus we have

$$y(t) = -\frac{d}{2}\left(\tan\frac{t}{2} - \frac{4}{\alpha^2}(\pi + t - \sin t)\right), \tag{33}$$

where $t \in (-\pi, t_0)$. Then the integral (24) can be computed as twice the integral along the positive y semiaxis. Thus the force is

$$\begin{aligned} F^x &= \frac{2\lambda}{\pi d}\int_{-\infty}^{u_0} \exp\left(\frac{u}{2\lambda}\right)\sqrt{\exp\left(-\frac{u}{2\lambda}\right) - 1}\, du \\ &= \frac{4\lambda^2}{\pi d}\int_{-\pi}^{t_0} \sin^2\frac{t}{2} dt = \frac{2\lambda^2}{\pi d}(\pi + t_0 - \sin t_0). \end{aligned} \tag{34}$$

In Eq. (34), u_0 is such that $y(u_0)=0$. In the Coulombian case ($b \to \infty$, i.e., $\alpha \to \infty$) $u_0=0$, so $t_0=0$ and the Coulomb force is recovered. Instead, in Born-Infeld theory, $u_0<0$ and $-\pi < t_0 < 0$. The dependence of the interaction on b is given through the value t_0 which depends on $\alpha = bd/\lambda$. Since $t_0 - \sin t_0$ is negative for $-\pi < t_0 < 0$, it is concluded that the interaction between equal monopoles is repulsive for all values of α, but less intense than the Coulombian interaction. In particular, for $\alpha \to 0$, $t_0 \to -\pi$; thus the repulsive interaction force vanishes for $d \to 0$.

In order to get an explicit correction to the Coulombian force we will take into account that t_0 is small for high values of α. Thus we can try to solve for t_0 in the equation $y(t_0) = 0$ [see Eq. (33)] by writing t_0 as a power series in α^{-2}. In this way we reach the result $t_0 = -8\pi\alpha^{-2} + 128\pi^3\alpha^{-6}/3 - 2048\pi^3\alpha^{-8}/3 + O(\alpha^{-10})$. Therefore the force (34) is

$$F^x \underset{\alpha\to\infty}{\longrightarrow} \frac{2\lambda^2}{d}\left(1 - \frac{256\pi^2}{3\alpha^6}\right) + O(\alpha^{-10}). \tag{35}$$

Thus, the Born-Infeld correction to the repulsive force between equal charges is very weak. Notice that d in (35) is not the real distance D between the charges. D goes to d when $b \to \infty$, but D is smaller than d for equal charges (see Fig. 3):

$$D_{\text{repulsive}} = d - \frac{2d}{\alpha^2}[\ln(\alpha^2) - 1] + O(\alpha^{-4}). \tag{36}$$

B. Opposite charges

We will now repeat the former steps for the case of the attractive charge configuration consisting of a charge $-\lambda$ at $x=-d/2$ and a charge λ at $x=d/2$. The complex Coulombian potential is

$$w = -2\lambda \ln\left(\frac{2z/d - 1}{2z/d + 1}\right). \tag{37}$$

Here we have chosen the integration constant such that $w(z)$ is null at infinity. Inverting (37), we obtain the Coulombian mapping

$$z = \frac{d}{2}\coth\frac{w}{4\lambda}. \tag{38}$$

This means that the function $p(w)$ is

$$p(w) = -\frac{d}{8\lambda}\sinh^{-2}\frac{w}{4\lambda}. \tag{39}$$

Thus the Born-Infeld mapping becomes

$$z = \frac{d}{2}\left[\coth\frac{w}{4\lambda} + \frac{4}{\alpha^2}\left(\frac{\bar{w}}{2\lambda} - \sinh\frac{\bar{w}}{2\lambda}\right)\right]. \tag{40}$$

The points where the electrostatic field reaches the value b [i.e., $|p(w)|=(2b)^{-1}$] belong to the curve

$$\frac{u}{2\lambda} = \pm \operatorname{arccosh}\left(\frac{\alpha}{2} + \cos\frac{v}{2\lambda}\right). \tag{41}$$

If $\alpha > 4$ this curve decomposes into two separate parts ("charges") at each side of the y axis. As in the previous case, the Born-Infeld mapping (40) fails to be periodic due to the presence of a linear term; so the u-v domain in mapping (40) should be properly restricted to get a single-valued potential $u(x,y)$. The branch of the complex potential w to be kept is the one matching the Coulombian potential at infinity. Differing from the previous case, this branch does reach the curves (41). Figures 4 and 5 show the equipotential (u = const) and field (v = const) lines surrounding the right charge, and their relation with the curve (41). Again the lines $v=0$ coincide with the piece of the x axis going between the charges and infinity. But in this case the domain of $|v|/(2\lambda)$ has to be extended beyond π to reach the piece of the x axis joining the charges. Both the potential u and the field **E** are discontinuous at the charge [the field attains the maximum value b at the exterior side of the charge, i.e., the side where $0 \le |v|/(2\lambda) \le \pi$]. In addition, the field is discontinuous at the branch cut on the x axis between the charges.

For $\alpha \le 4$ the curve of maximum field becomes closed, as is typical for a dipole [21]. In this case, the "charge" distri-

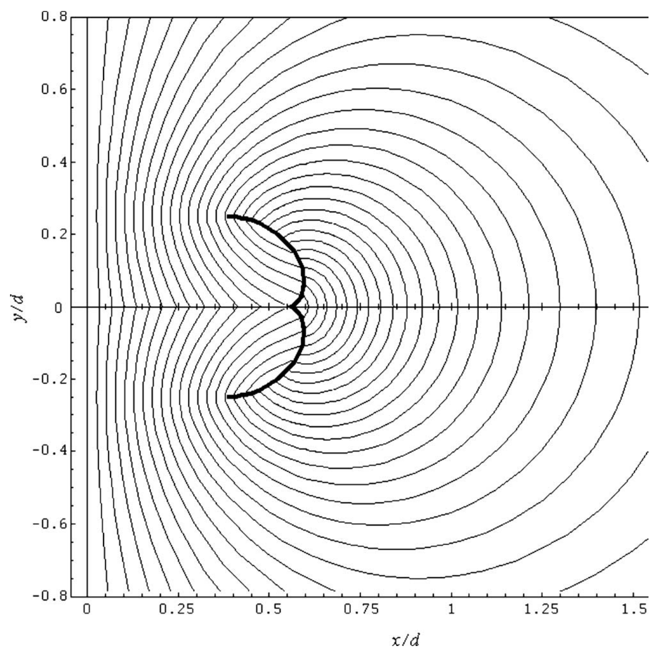

FIG. 4. Born-Infeld equipotential lines (u=const) for α=5, together with the curve describing the right charge.

bution becomes a unique object; the Coulombian zone $|p(w)|>(2b)^{-1}$ is, of course, the outside of the object. Figure 6 shows the curves $|p(w)|=(2b)^{-1}$ for different values of α. The curves display cusps on the x axis ($v=0$) at $|x|>d/2$. At the cusps, $u=\pm 2\lambda\,\text{arccosh}(\alpha/2+1)$.

We will calculate the force (25) for $\alpha>4$, which is the case where the charges are separate. Since the y axis is characterized by $u=0$, then Eq. (40) indicates that the positive y axis accepts a parametrization similar to (33) whenever the parameter t is defined as $t=v/(2\lambda)-\pi$:

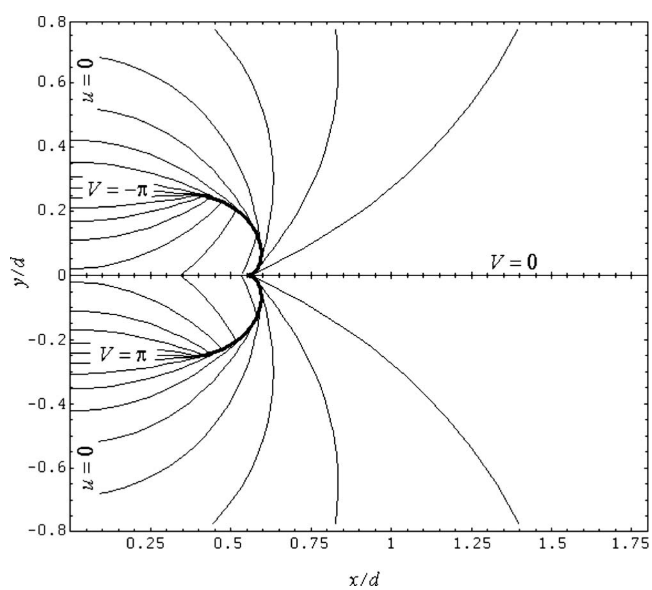

FIG. 5. Born-Infeld field lines (v=const) for α=5, together with the curve describing the right charge. In the figure V stands for $v/(2\lambda)$.

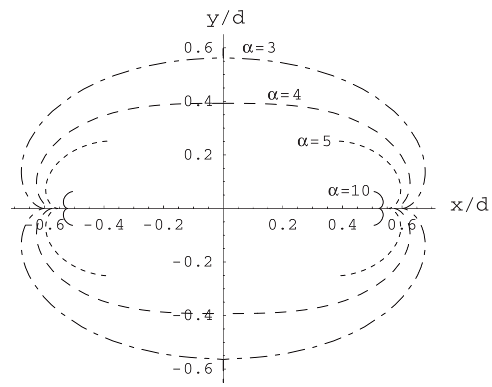

FIG. 6. Born-Infeld opposite "charges" for α=10 and 5. For $\alpha\leq 4$ the charges merge into a single object.

$$y(t)=-\frac{d}{2}\left(\tan\frac{t}{2}-\frac{4}{\alpha^2}(\pi+t+\sin t)\right). \tag{42}$$

This function is monotonic for $t\in(-\pi,t_0)$, where $0<t_0<\pi$ is the parameter satisfying $y(t_0)=0$. Since $p(w)|_{y\text{ axis}}=d/(8\lambda)\sin^{-2}[v/(4\lambda)]$, then the force (25) is

$$F^x=-\frac{4\lambda^2}{\pi d}\int_{-\pi}^{t_0}\cos^2\frac{t}{2}dt=-\frac{2\lambda^2}{\pi d}(\pi+t_0+\sin t_0). \tag{43}$$

For $b\to\infty$ ($\alpha\to\infty$), $t_0\to 0$ in Eq. (43), so the Coulombian force is recovered. Since $t_0\in(0,\pi)$, it is concluded that the attraction between opposite monopoles is more intense than the Coulombian interaction. We will solve for t_0 in the equation $y(t_0)=0$ [see Eq. (42)] by writing t_0 as a power series in $(\alpha^2-16)^{-1}$. The result is $t_0=8\pi(\alpha^2-16)^{-1}[1-16\pi^2(\alpha^2-16)^{-2}/3]+O[(\alpha^2-16)^{-4}]$. Therefore the Born-Infeld interaction (43) between opposite monopoles behaves as

$$F^x \underset{\alpha\to\infty}{\longrightarrow} \frac{2\lambda^2}{d}\left(1+\frac{16}{\alpha^2-16}-\frac{512\pi^2}{3(\alpha^2-16)^3}\right)+O[(\alpha^2-16)^{-4}]. \tag{44}$$

Differing from the repulsive case, the attractive interaction receives a more perceptible correction of order $(\alpha^2-16)^{-1}$. Notice that d in (44) is not the distance D between the cusps in Fig. 6. By computing the positions of the cusps for opposite charges, we see that D is larger than d:

$$D_{\text{attractive}}=d+\frac{2d}{\alpha^2-16}[\ln(\alpha^2-16)-3]+O[(\alpha^2-16)^{-3/2}]. \tag{45}$$

V. CONCLUSIONS

The first goal of Born-Infeld theory was to obtain a pointlike charge solution with finite self-energy. However, this solution has intriguing features: $\partial L_{\text{BI}}/\partial\mathbf{E}$ still diverges, and $\mathbf{E}$ is finite at the charge position (an unpleasant property for a vector field at its center of symmetry). However, these disagreeable features do not cause any trouble to the interaction between charges. We have considered a two-charge field (which differs from the mere superposition of two one-charge fields, since the theory is nonlinear). To compute the interaction between parts of this field configuration one must consider the momentum flux through a surface separating both subsystems. According to the method developed in Ref. [21], the interaction force is given by expressions (17) and (18), where p is a Coulombian function, and the Born-Infeld features are encoded in the integration interval. Although we have obtained the force in a parametric form [the parameter t_0 in forces (34) and (43) comes from the transcendental equation $y(t_0)=0$ in (33) and (42), respectively], we have succeeded in computing Born-Infeld corrections to Coulombian interactions. In addition, we have proved that the interaction force between equal charges is well behaved and goes to zero when the charges approach each other. This limit cannot be reached for opposite charges because they merge in a unique dipolar object of finite size.

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[22] Notice that the dependence of the Lagrangian on S and P in Eq. (1) varies with the dimension. Expression (1) is the result for 3+1 dimensions. Instead, $\det(g_{ij}+b^{-1}F_{ij})=1+b^{-2}2S$ in 2+1 dimensions. Electrostatic solutions have $P=0$, so the difference is irrelevant in this case. Therefore, our results can be regarded both as describing the Born-Infeld electrostatic interaction between pointlike charges in 2+1 dimensions or parallel charged lines in 3+1 dimensions.